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Analytic versus numerical approaches: the two-wire transmission line

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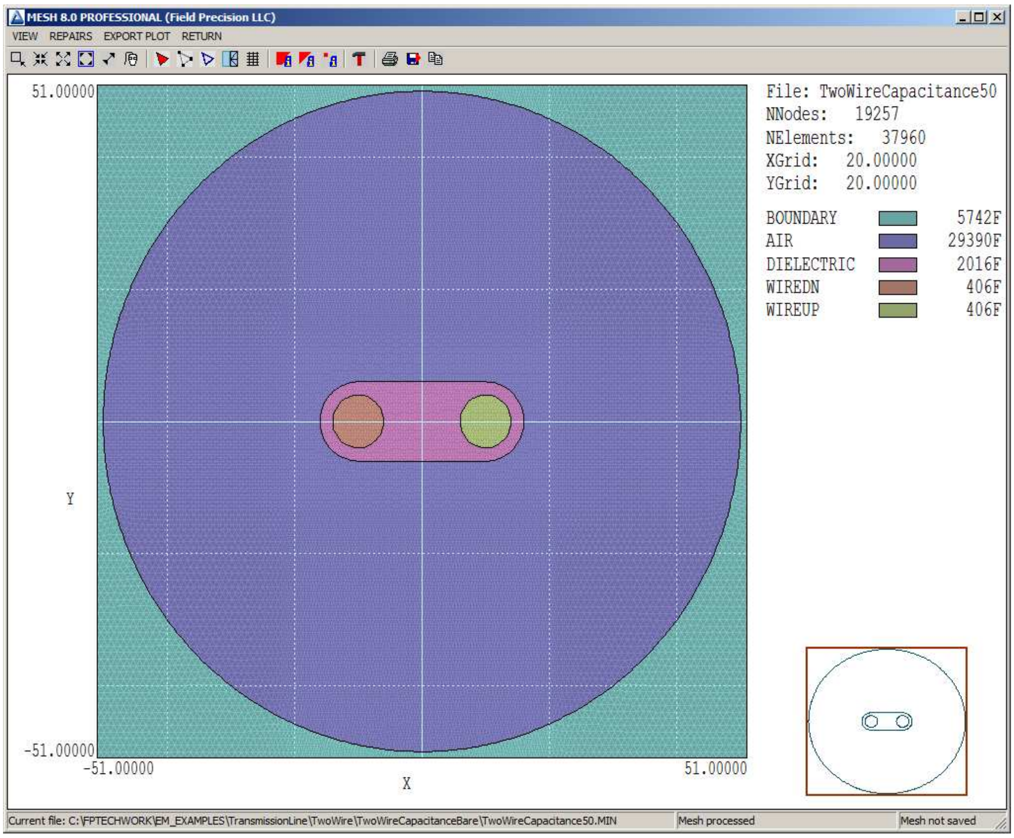

Figure 1: Mesh for the electrostatic calculation with 50.0 mm solution boundary radius.

This report describes calculations of a familiar example in electrostatics and magnetostatics, the capacitance and inductance per length of a two wire transmission line. Despite its simplicity, the example provides an opportunity to consider the relative merits of analytic versus numerical methods. To review, the following equations hold for bare circular wires in free space with radius a separated by distance D:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(D/2a)}. \quad \text{(F/m)} \tag{1}$$

$$C = \pi\mu_0 \cosh^{-1}(D/2a). \quad \text{(H/m)} \tag{2}$$

We will use test parameters $a = 4.0$ mm and $D = 20.0$ mm. If you don't own an HP35s, there are many online function evaluation sites to determine that $\cosh^{-1}(5) = 1.56678$. The theoretical values are $C = 1.775 \times 10^{-11}$ (F/m) and $L = 6.267^{-7}$ (H/m).

For the comparison numerical calculations, I used the two-dimensional **EStat** code for the electric field calculation and **PerMag** for the magnetic field. Figure 1 shows the geometry for the electrostatic solution. In order of definition, the regions are:

1. Fixed potential elements representing a grounded boundary initially filling the solution volume.
2. Air elements to cover a circle of radius $R = 50.0$ mm.
3. Dielectric elements representing the support structure for the transmission line. The condition $\epsilon_r = 1.0$ is used to represent bare wires.
4. Elements of the lower wire with fixed potential -0.5v.
5. Elements of the upper wire with fixed potential +0.5v.

In principle, the boundary radius R should be infinite for a comparison with the analytic results. In practice, finite-element solutions are performed over finite volumes with defined boundaries. In this case, I have chosen a relatively large boundary with coarser elements at a distance from the transmission line. The element resolution at the wire is 0.5 mm. With this value, area of wire elements calculated by the code corresponds to an effective radius of $a = 3.9948$ mm.

The code can perform an integral of field energy per length (U) by summing over all elements and organizing the result by region number. For the choice of a 1.0 V potential difference, the capacitance per length between the wires is $C = 2U$. The code result is $C = 1.852 \times 10^{-11}$ F/m, about 4.3% higher than the infinite-space analytic value. The question is whether the difference results from the finite boundary or from an accuracy limitation of the numerical method. One way to address the issue is to perform a set of solutions with different boundary radii. This operation is relatively easy with **Mesh** and **EStat**. To create a new **Mesh** input file, the original solution is loaded into the *Mesh Drawing Editor* where the boundary circle of the air region is deleted and replaced with a smaller circle. The automatic task feature of **EStat** can them be used to run and to analyze the additional solutions as a batch process. Figure 2 shows the results. In cases where a quantity such as the boundary radius goes to infinity, it is better to make a plot of the inverse of the quantity so that one of the limits is 0.0. Accordingly, the figure shows the calculated capacitance versus the inverse of the boundary radius. The boundary effect is clearly significant, and it is likely that the calculated cacitance would approach the theoretical value with a larger solution volume.

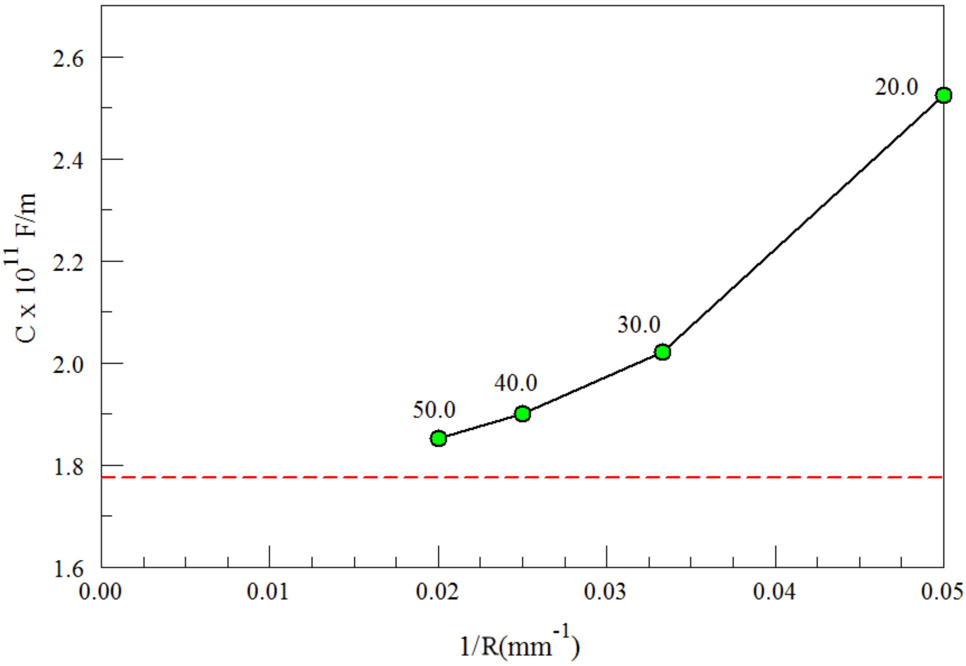

Figure 2: Capacitance per length between the wires as a function of the inverse of the boundary radius. The dashed line shows the theoretical value.

The magnetic field calculation presents a challenge. In magnetic field codes, the standard input is a specified current or current density distributed over the solution volume. In the case of a high-frequency transmission line, the current is confined over a thin layer on the wire surface. The surface current density is not distributed uniformly, but rather in a way to maintain the condition that the normal component of **B** at the surface is zero. Thus there are two potential problems for a finite-element code:

- Very small elements would be required to represent a surface current.
- The distribution of current over the surface is not known in advance.

Fortunately, there is simple solution. We can let the code do the work of meeting the boundary conditions. Figure $3a$ shows the approach. As before, there is an inner air volume ($\mu_r = 1.0$) with an outer boundary region at fixed potential ($A_z = 0.0$). Each wire has an inner drive region carrying ± 1.0 A surrounded by a sheath has very small relative permeability, $\mu_r \ll 1.0$. The condition on a boundary between media with widely different μ_r is $B_\perp = 0.0$. Therefore, the effective surface current density will have the correct distribution for a drive current placed anywhere inside the sheath[1]. For the code calculation, the value $\mu_r = 0.001$ gave a good boundary distribution without impeding solution convergence.

The inductance can be determined from a field energy integral, $L = 2U$. Here, we must include only the energy integral from the air region because there is non-zero energy in the two drive regions. Figure 4 shows the results with a fitted curve passing through the theoretical value at the origin. As with the capacitance calculation, the absolute values are consistent with theory and the effects of the finite boundary are significant. One conclusion is that we must pay attention to boundaries in finite-element field calculations. Comparisons with ideal solutions in infinite space present challenges.

On the other hand, the real world is seldom ideal and often boundaries are a virtue. For example, suppose the transmission line passes through a grommet in a shield wall or through a pipe. This case is easy to handle with direct numerical methods but difficult to approach analytically, especially if the line is offset in the channel. An *exact* solution would probably involve extended series solutions. One calculation goal may be to estimate the impedance mismatch of a feed-through. For the example, we can use the capacitance and inductance per length to determine the impedance of the two-wire line, $Z = \sqrt{L/C}$ (Ω). Figure 5 shows the results. There is a significant mismatch, even with a large pipe radius. This is the reason that coaxial transmission lines are preferred, even though they are more expensive to fabricate. Other geometries (*e.g.* non-circular pipe, displaced line, line above a ground plane, line encased in a dielectric support,...) are easy to handle with numerical codes, but would be almost impossible to solve analytically.

To conclude, the question is when to use numerical methods rather seek closed-form solutions. The relative advantages depend on the context:

- Learning electromagnetism.
- Using electromagnetism for applications.

Although I create them, I feel there is little role for numerical codes in electromagnetic education. After many years teaching fields and waves, I believe that the old-fashioned way is the best, even if it involves Smith charts. The mathematical techniques and physical insights gained learning the subject analytically are invaluable. In education, there are important differences between the experience of using a good textbook and a numerical code:

[1] If you are not convinced, set up a solution with a single circular sheath at the origin. For any position of the internal drive current I, the magnetic flux density outside the sheath is azimuthally symmetric with magnitude $B_\theta = \mu_0 I/2\pi r$.

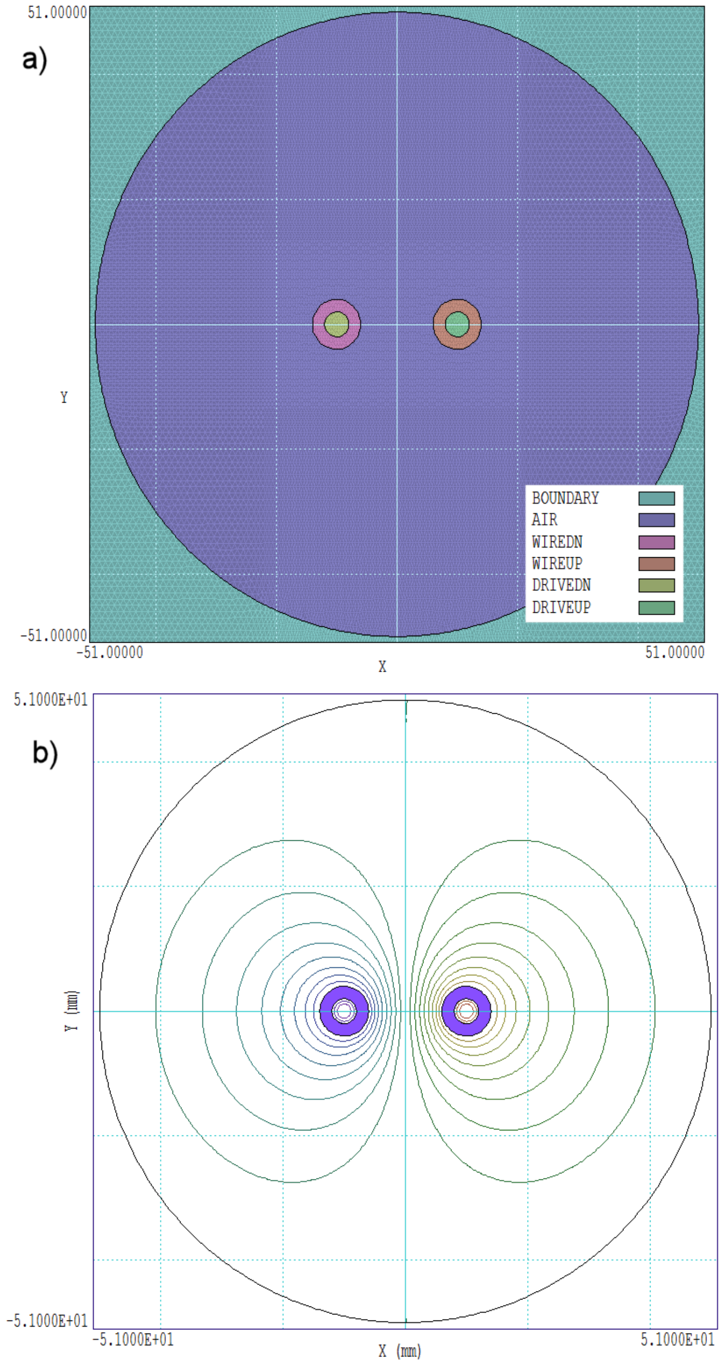

Figure 3: Magnetic field calculation. a) Mesh with 50.0 mm solution boundary radius. b) Lines of magnetic field induction, with the sheaths ($\mu_r = 0.001$) highlighted.

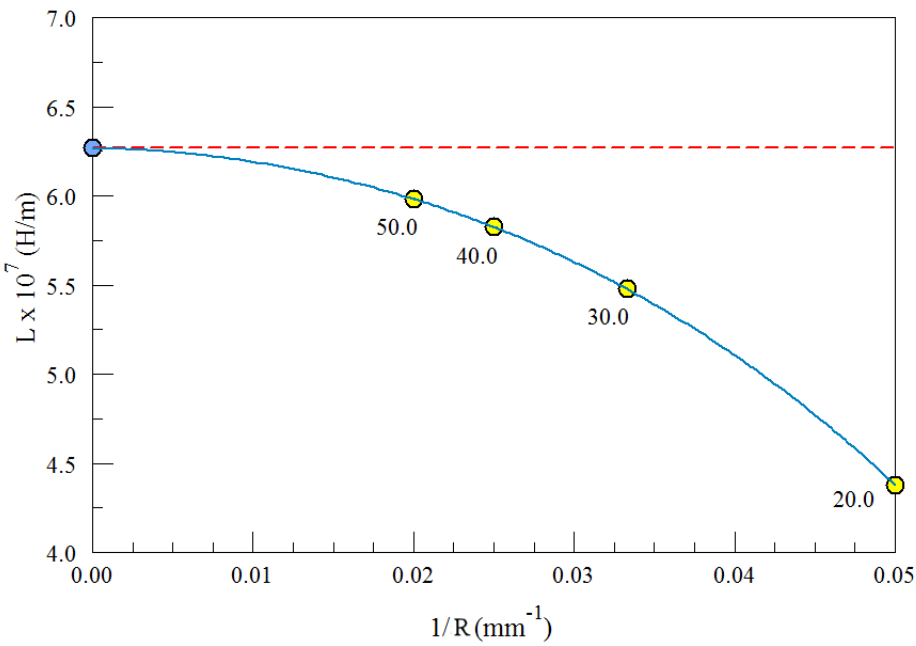

Figure 4: Inductance per length between the wires as a function of the inverse of the boundary radius. The dashed line shows the theoretical value. The solid blue line is a polynomial fit passing through the theoretical value at the origin.

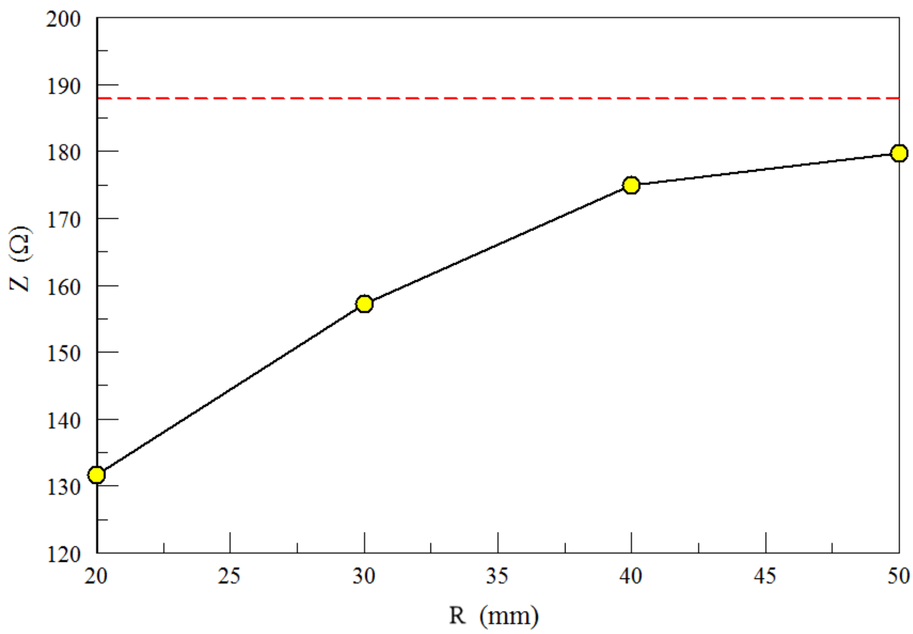

Figure 5: Impedance of a two-wire passing through a pipe of radius R. The dashed line shows the line impedance in free space.

- In a textbook, students are exposed to a variety of critical concepts with broad application to a scientific education. In contrast, with a code students would occupy their time learning a specific interface with little general applicability.

- Textbook results are the ultimate open-source material. The notations and derivations created over more than a century are transparent to the user. Only the best approaches and most valid results can survive. In contrast, three-dimensional numerical codes are commercial ventures with proprietary content. Often, such codes are not ranked by their accuracy or speed, but rather by how intensely they are marketed.

On the other hand, numerical codes are invaluable application tools for a user with a solid knowledge of electromagnetic theory. Even in the simple example of the two-wire transmission line, we can conceive many variants of geometry or material properties beyond symmetric bare wires. An analytic solution of a problem that might require an hour of code set up could demand enough work to constitute of Ph.D. thesis.